Basic Mathematics

Introduction to Complex Numbers

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The aim of this package is to provide a short study and self assessment programme for students who wish to become more familiar with complex numbers.

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1. The Square Root of Minus One!

If we want to calculate the square root of a negative number, it rapidly becomes clear that neither a positive or a negative number can do it.

$$\text{E.g.,} \qquad \sqrt{-1} \neq \pm 1, \text{ since} \qquad 1^2 = (-1)^2 = +1\,.$$

To find $\sqrt{-1}$ we introduce a new quantity, i, defined to be such that $i^2 = -1$. (Note that engineers often use the notation j.)

Example 1

$$\begin{array}{rrcl} \text{(a)} & \sqrt{-25} & = & 5i \\ & \text{Since } (5i)^2 & = & 5^2 \times i^2 \\ & & = & 25 \times (-1) \\ & & = & -25\,. \end{array}$$

$$
\begin{aligned}
\text{(b)} \qquad \sqrt{-\frac{16}{9}} &= \frac{4}{3}i \\
\text{Since } (\frac{4}{3}i)^2 &= \frac{16}{9} \times (i^2) \\
&= -\frac{16}{9}\,.
\end{aligned}
$$

2. Real, Imaginary and Complex Numbers

Real numbers are the usual positive and negative numbers.

If we multiply a real number by i, we call the result an *imaginary* number. Examples of imaginary numbers are: i, $3i$ and $-i/2$.

If we add or subtract a real number and an imaginary number, the result is a *complex* number. We write a complex number as

$$z = a + ib$$

where a and b are real numbers.

3. Adding and Subtracting Complex Numbers

If we want to *add* or *subtract* two complex numbers, $z_1 = a + ib$ and $z_2 = c+id$, the rule is to add the real and imaginary parts separately:

$$\begin{aligned} z_1 + z_2 &= a + ib + c + id = a + c + i(b + d) \\ z_1 - z_2 &= a + ib - c - id = a - c + i(b - d) \end{aligned}$$

Example 2

$$\begin{aligned} (a) \qquad (1 + i) + (3 + i) &= 1 + 3 + i(1 + 1) = 4 + 2i \\ (b) \quad (2 + 5i) - (1 - 4i) &= 2 + 5i - 1 + 4i = 1 + 9i \end{aligned}$$

Exercise 1. Add or subtract the following complex numbers. (Click on the green letters for the solutions.)

(a) $(3 + 2i) + (3 + i)$ (b) $(4 - 2i) - (3 - 2i)$

(c) $(-1 + 3i) + \frac{1}{2}(2 + 2i)$ (d) $\frac{1}{3}(4 - 5i) - \frac{1}{6}(8 - 2i)$

Quiz To which of the following does the expression

$$(4 - 3i) + (2 + 5i)$$

simplify?

(a) $6 - 8i$ (b) $6 + 2i$
(c) $6 + 8i$ (d) $9 - i$

Quiz To which of the following does the expression

$$(3 - i) - (2 - 6i)$$

simplify?

(a) $3 - 9i$ (b) $1 - 7i$
(c) $1 + 5i$ (d) $1 + 5i$

4. Multiplying Complex Numbers

We *multiply* two complex numbers just as we would multiply expressions of the form $(x + y)$ together (see the package on Brackets)

$$\begin{aligned}(a+ib)(c+id) &= ac + a(id) + (ib)c + (ib)(id) \\ &= ac + iad + ibc - bd \\ &= ac - bd + i(ad + bc)\end{aligned}$$

Example 3

$$\begin{aligned}(2+3i)(3+2i) &= 2\times 3 + 2\times 2i + 3i\times 3 + 3i\times 2i \\ &= 6 + 4i + 9i - 6 \\ &= 13i\end{aligned}$$

Exercise 2. Multiply the following complex numbers. (Click on the green letters for the solutions.)

(a) $(3+2i)(3+i)$

(b) $(4-2i)(3-2i)$

(c) $(-1+3i)(2+2i)$

(d) $(4-5i)(8-2i)$

Quiz To which of the following does the expression

$$(2-i)(3+4i)$$

simplify?

(a) $5+4i$

(b) $6+11i$

(c) $10+5i$

(d) $6+i$

5. Complex Conjugation

For any complex number, $z = a + ib$, we *define* the complex conjugate to be: $z^* = a - ib$. It is very useful since:

$$\begin{aligned} z + z^* &= a + ib + (a - ib) = 2a \\ zz^* &= (a + ib)(a - ib) = a^2 + iab - iab - a^2 - (ib)^2 = a^2 + b^2 \end{aligned}$$

The *modulus* of a complex number is defined as: $|z| = \sqrt{zz^*}$

Exercise 3. Combine the following complex numbers and their conjugates. (Click on the green letters for the solutions.)

(a)	If $z = (3 + 2i)$, find $z + z^*$	(b)	If $z = (3 - 2i)$, find zz^*
(c)	If $z = (-1 + 3i)$, find zz^*	(d)	If $z = (4 - 5i)$, find $\|z\|$

Quiz Which of the following is the modulus of

$$4 - 2i?$$

(a) $5 + 4i$ (b) $6 + 11i$
(c) $10 + 5i$ (d) $6 + i$

6. Dividing Complex Numbers

The *trick* for dividing two complex numbers is to multiply top and bottom by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{z_2^*}{z_2^*} = \frac{z_1 z_2^*}{z_2 z_2^*}$$

The denominator, $z_2 z_2^*$, is now a real number.

Example 4

$$\begin{aligned} \frac{1}{i} &= \frac{1}{i} \times \frac{-i}{-i} \\ &= \frac{-i}{i \times (-i)} \\ &= \frac{-i}{1} \\ &= -i \end{aligned}$$

Example 5

$$
\begin{aligned}
\frac{(2+3i)}{(1+2i)} &= \frac{(2+3i)}{(1+2i)}\frac{(1-2i)}{(1-2i)} \\
&= \frac{(2+3i)(1-2i)}{1+4} \\
&= \frac{1}{5}(2+3i)(1-2i) \\
&= \frac{1}{5}(2-4i+3i+6) = \frac{1}{5}(8-i)
\end{aligned}
$$

Exercise 4. Perform the following division: (Click on the green letters for the solutions.)

(a) $\dfrac{(2+4i)}{i}$ (b) $\dfrac{(-2+6i)}{(1+2i)}$

(c) $\dfrac{(1+3i)}{(2+i)}$ (d) $\dfrac{(3+2i)}{(3+i)}$

Quiz To which of the following does the expression

$$\frac{8-i}{2+i}$$

simplify?

(a) $3-2i$
(b) $2+3i$
(c) $4-\frac{1}{2}i$
(d) 4

Quiz To which of the following does the expression

$$\frac{-2+i}{2+i}$$

simplify?

(a) -1
(b) $\frac{1}{5}(-5+7i)$
(c) $-1+\dfrac{1}{2}i$
(d) $\frac{1}{5}(-3+4i)$

7. Quiz on Complex Numbers

Begin Quiz In each of the following, simplify the expression and choose the solution from the options given.

1. $(3 + 4i) - (2 - 3i)$

(a) $3 - i$ (b) $5 + 7i$
(c) $1 + 7i$ (d) $1 - i$

2. $(3 + 3i)(2 - 3i)$

(a) $6 - 8i$ (b) $6 + 8i$
(c) $-3 + 3i$ (d) $15 - 3i$

3. $|12 - 5i|$

(a) 13 (b) $\sqrt{7}$
(c) $\sqrt{119}$ (d) -12.5

4. $(13 - 17i)/(5 - i)$

(a) $\frac{13}{5} + 17i$ (b) $3 + i$
(c) $3 + 2i$ (d) $2 - 3i$

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a)

$$\begin{aligned} (3+2i)+(3+i) &= 3+2i+3+i \\ &= 3+3+2i+2i \\ 6+3i & \end{aligned}$$

Click on the green square to return

Exercise 1(b) Here we need to be careful with the signs!

$$\begin{aligned} 4 - 2i - (3 - 2i) &= 4 - 2i - 3 + 2i \\ &= 4 - 3 - 2i + 2i \\ &= 1 \end{aligned}$$

A purely real result

Click on the green square to return

□

Exercise 1(c) The factor of $\frac{1}{2}$ multiplies both terms in the complex number.

$$\begin{aligned} -1+3i+\frac{1}{2}(2+2i) &= -1+3i+1+i \\ &= 4i \end{aligned}$$

A purely imaginary result.

Click on the green square to return □

Exercise 1(d)

$$\begin{aligned}
\frac{1}{3}(2-5i)-\frac{1}{6}(8-2i) &= \frac{2}{3}-\frac{5}{3}i-\frac{8}{6}+\frac{2}{6}i \\
&= \frac{2}{3}-\frac{5}{3}i-\frac{4}{3}+\frac{1}{3}i \\
&= \frac{2}{3}-\frac{4}{3}-\frac{5}{3}i+\frac{1}{3}i \\
&= -\frac{2}{3}-\frac{4}{3}i
\end{aligned}$$

which we could also write as $-\frac{2}{3}(1+2i)$.

Click on the green square to return □

Exercise 2(a)

$$\begin{aligned}(3+2i)(3+i) &= 3\times 3+3\times i+2i\times 3+2i\times i\\ &= 9+3i+6i-2\\ &= 9-2+3i+6i\\ &= 7+9i\end{aligned}$$

Click on the green square to return $\square$

Exercise 2(b)

$$\begin{aligned}(4-2i)(3-2i) &= 4\times 3+4\times(-2i)-2i\times 3-2i\times -2i\\ &= 12-8i-6i-4\\ &= 12-4-8i-6i\\ &= 8-14i\end{aligned}$$

Click on the green square to return

Exercise 2(c)

$$\begin{aligned}(-1+3i)(2+2i) &= -1\times 2 - 1\times 2i + 3i\times 2 + 3i\times 2i\\ &= -2-2i+6i-6\\ &= -2-6-2i+6i\\ &= -8+4i\end{aligned}$$

Click on the green square to return $\square$

Exercise 2(d)

$$\begin{aligned}(2-5i)(8-3i) &= 2\times 8+2\times(-3i)-5i\times 8-5i\times(-3i)\\ &= 16-6i-40i-15\\ &= 16-15-6i-40i\\ &= 1-46i\end{aligned}$$

Click on the green square to return $\square$

Exercise 3(a)

$$\begin{aligned}
(3+2i)+(3+2i)^* &= \\
(3+2i)+(3-2i) &= 3+2i+3-2i) \\
&= 3+3+2i-2i \\
&= 6
\end{aligned}$$

Click on the green square to return

Exercise 3(b)

$$\begin{aligned}(3-2i)(3-2i)^* &= (3-2i)(3+2i)\\ &= 9+6i-6i-2i\times(2i)\\ &= 9-4i^2\\ &= 9+4=13\end{aligned}$$

Click on the green square to return

Exercise 3(c)

$$\begin{aligned}
(-1+3i)(-1+3i)^* &= (-1+3i)(-1-3i) \\
&= (-1)\times(-1)+(-1)(-3i)+3i(-1)+3i(-3i)) \\
&= 1+3i-3i-9i^2 \\
&= 1+9=10
\end{aligned}$$

Click on the green square to return □

Exercise 3(d)

$$\begin{aligned}
\sqrt{(4-3i)(4+3i)} &= \sqrt{4^2+4\times 3i-3i\times 4-3i\times 3i}\\
&= \sqrt{16+12i-12i-9i^2}\\
&= \sqrt{16+9}\\
&= \sqrt{25}=5
\end{aligned}$$

Click on the green square to return □

Exercise 4(a)

$$\begin{aligned}
\frac{(2+4i)}{i} &= \frac{(2+4i)}{i} \times \frac{-i}{-i} \\
&= \frac{(2+4i) \times (-i)}{+1} \\
&= (2+4i)(-i) \\
&= -2i - 4i^2 \\
&= 4 - 2i
\end{aligned}$$

Click on the green square to return

Exercise 4(b)

$$
\begin{aligned}
\frac{(-2+6i)}{(1+2i)} &= \frac{(-2+6i)}{(1+2i)} \times \frac{(1-2i)}{(1-2i)} \\
&= \frac{(-2+6i)(1-2i)}{1+4} \\
&= \frac{1}{5}(-2+6i)(1-2i) \\
&= \frac{1}{5}(-2+4i+6i-12i^2) \\
&= \frac{1}{5}(-2+10i+12) \\
&= \frac{1}{5}(-2+12+10i) \\
&= \frac{1}{5}(10+10i) = 2+2i
\end{aligned}
$$

Click on the green square to return

Exercise 4(c)

$$\begin{aligned}
\frac{(1+3i)}{(2+i)} &= \frac{(1+3i)}{(2+i)} \times \frac{(2-i)}{(2-i)} \\
&= \frac{(1+3i)(2-i)}{4+1} \\
&= \frac{1}{5}(2-i+6i-3i^2) \\
&= \frac{1}{5}(2+3+5i) \\
&= \frac{1}{5}(5+5i) = 1+i
\end{aligned}$$

Click on the green square to return □

Exercise 4(d)

$$
\begin{aligned}
\frac{(3+2i)}{(3+i)} &= \frac{(3+2i)}{(3+i)} \times \frac{(3-i)}{(3-i)} \\
&= \frac{(3+2i)(3-i)}{9+1} \\
&= \frac{1}{10}(3+2i)(3-i) \\
&= \frac{1}{10}(9-3i+6i-2i^2) \\
&= \frac{1}{10}(9+2+3i) \\
&= \frac{1}{10}(11+3i)
\end{aligned}
$$

Click on the green square to return $\square$

Solutions to Quizzes

Solution to Quiz:

$$
\begin{aligned}
(4-3i)+(2+5i) &= 4-3i+2+5i \\
&= 4+2-3i+5i \\
&= 6+2i
\end{aligned}
$$

End Quiz

Solution to Quiz:

$$\begin{aligned}(2-i)(3+4i) &= 2\times 3+2\times(4i)-i\times 3-i\times(4i)\\ &= 6+8i-3i-4i^2\\ &= 6+5i+4\\ &= 10+5i\end{aligned}$$

End Quiz

Solution to Quiz:

$$\begin{aligned}(2-i)(3+4i) &= 2\times 3+2\times(4i)-i\times 3-i\times(4i)\\ &= 6+8i-3i-4i^2\\ &= 6+5i+4\\ &= 10+5i\end{aligned}$$

End Quiz

Solution to Quiz:

$$\begin{aligned}(2-i)(3+4i) &= 2\times 3+2\times(4i)-i\times 3-i\times(4i)\\ &= 6+8i-3i-4i^2\\ &= 6+5i+4\\ &= 10+5i\end{aligned}$$

End Quiz

Solution to Quiz:

$$
\begin{aligned}
\frac{8-i}{2+i} &= \frac{8-i}{2+i}\,\frac{2-i}{2-i} \\
&= \frac{(8-i)(2-i)}{2^2+1^2} \\
&= \frac{(8\times 2+8\times(-i)-i\times 2-i\times(-i))}{5} \\
&= \frac{1}{5}\,(16-8i-2i-1) \\
&= \frac{1}{5}\,(15-10i) = 3-2i
\end{aligned}
$$

End Quiz

Solution to Quiz:

$$
\begin{aligned}
\frac{-2+i}{2+i} &= \frac{-2+i}{2+i}\,\frac{2-i}{2-i} \\
&= \frac{(-2+i)(2-i)}{2^2+1^2} \\
&= \frac{1}{5}\left(-2\times 2 - 2\times(-i) + i\times 2 + i\times(-i)\right) \\
&= \frac{1}{5}\left(-4+2i+2i+1\right) \\
&= \frac{1}{5}\left(-3+4i\right)
\end{aligned}
$$

End Quiz